Lecture III (EKC) Koszul connections

In the previous lecture we defined the notion of an Ehresmann connection on a ppal G-bundle $P \to M$. We had three equivalent characterisations of such a connection:

① a G-invariant horizontal distribution $\mathcal{H}$:
$$T_pP = \ker(\pi_*)_p \oplus \mathcal{H}_p \quad \text{and} \quad (r_g)_*\mathcal{H}_p = \mathcal{H}_{p\cdot g}$$

② a $\mathfrak{g}$-valued 1-form $\omega \in \Omega^1(P;\mathfrak{g})$ obeying $r_g^*\omega = \mathrm{Ad}_{g^{-1}} \circ \omega$

& ③ a family $\{A_\alpha \in \Omega^1(U_\alpha;\mathfrak{g})\}$ obeying $A_\alpha = \mathrm{Ad}_{g_{\alpha\beta}} \circ A_\beta + g_{\beta\alpha}^*\theta$ on $U_{\alpha\beta}$ $(\neq \emptyset)$.

transition function $g_{\alpha\beta} : U_{\alpha\beta} \to G$

LI MC form

We shall return to this later when we introduce Cartan connections, but today's lecture is about the Koszul connections on associated vector bundles to a PFB. If PFBs describe "gauge fields", their associated VBs describe "matter fields". We start by defining the objects of interest.

19. __Definition__ A real (resp. complex) rank-k vector bundle $E \xrightarrow{\pi} M$ is a fibre bundle whose fibres are k-dimensional real (resp. complex) vector spaces and whose local trivialisations $\psi : \pi^{-1}U \to U \times \mathbb{R}^k$ (resp. $\psi : \pi^{-1}U \to U \times \mathbb{C}^k$) restrict fibrewise to isomorphisms $\psi : E_a \to \{a\} \times \mathbb{R}^k$ (resp. $\psi : E_a \to \{a\} \times \mathbb{C}^k$) of real (resp. complex) vector spaces.

We will show that with a ppal G-bundle $P \xrightarrow{\pi} M$ and a finite-dimensional (real or complex) representation $\rho : G \to GL(V)$, there is associated a vector bundle $E \xrightarrow{\varpi} M$, and that every vector bundle can be obtained in that way.

Let $P \xrightarrow{\pi} M$ be a ppal G-bundle and let $\rho : G \to GL(V)$ be a Lie group homomorphism (ie: a representation of G) where V is a finite-dimensional vector space. Since G acts freely on P, it also acts freely on $P \times V$ via the right action

$$(p, v) \cdot g = (p \cdot g, \rho(g^{-1}) \cdot v)$$

We let $E := P \times_G V$ denote the quotient $(P \times V)/G$ via the above action. It is the total space of a vector bundle $E \xrightarrow{\varpi} M$, where $\varpi : P \times_G V \to M$ is such that $\varpi([(p,v)]) = \pi(p)$, which is well-defined because $\pi(p \cdot g) = \pi(p)$. It is called an associated* vector bundle to the PFB $P \to M$.

* associated with P via the representation ρ.

We can understand this bundle via its local construction.

Let $\{(U_\alpha, \psi_\alpha)\}_{\alpha \in A}$ be a trivialising atlas for P with transition functions $\{g_{\alpha\beta} : U_{\alpha\beta} \to G\}$ obeying the cocycle conditions. Then we may trivialise $P \times_G V$ on each U_α (since $(U_\alpha \times G \times V)/G \simeq U_\alpha \times V$) and the transition functions are $\{\rho \circ g_{\alpha\beta} : U_{\alpha\beta} \to GL(V)\}$. More concretely we define $P \times_G V := \bigsqcup_\alpha (U_\alpha \times V)/\sim$ where $(a, v) \sim (a, \rho(g_{\alpha\beta}(a)) \cdot v)$ for all $a \in U_{\alpha\beta}$. We use left-multiplication to ensure G-equivariance with G acting on the right on P.

Let $P \xrightarrow{\pi} M$ be a G-PFB and $E := P \times_G V \xrightarrow{\varpi} M$ an associated VB with $\rho : G \to GL(V)$. Let $\Gamma(E) = \{ s : M \to E \mid \varpi \circ s = id_M \}$ denote the $C^\infty(M)$-module of sections of E. Let $C^\infty_G(P, V) = \{ \xi : P \to V \mid r_g^* \xi = \rho(g)^{-1} \circ \xi \ \forall g \in G \}$ be the G-equivariant functions $P \to V$. We give $C^\infty_G(P,V)$ the structure of a $C^\infty(M)$-mod by declaring $f \cdot \xi = \pi^* f \, \xi \quad \forall f \in C^\infty(M)$ and $\xi \in C^\infty_G(P,V)$.

<u>20. Proposition</u> There is a $C^\infty(M)$-module isomorphism

$$\Gamma(E) \cong C^\infty_G(P,V).$$

Proof Let $\sigma \in \Gamma(E)$. Let $\Psi_\alpha : \varpi^{-1}U_\alpha \to U_\alpha \times V$ be a local trivialisation and define $\sigma_\alpha : U_\alpha \to V$ by $(\Psi_\alpha \circ \sigma)(a) = (a, \sigma_\alpha(a))$. On overlaps $U_{\alpha\beta}$, the local functions σ_α and σ_β are related by $\sigma_\alpha(a) = \rho(g_{\alpha\beta}(a))\sigma_\beta(a)$ $\forall a \in U_{\alpha\beta}$, where $g_{\alpha\beta} : U_{\alpha\beta} \to G$ are the transition functions of $P \to M$.
Given $\sigma \in \Gamma(E)$ we define $\xi_\alpha : \pi^{-1}U_\alpha \to V$ by $\xi_\alpha((\pi^* s_\alpha)(p)) = \sigma_\alpha(\pi(p))$ and extend by $\xi_\alpha((\pi^* s_\alpha)(p)\cdot g) = \rho(g)^{-1} \circ \sigma_\alpha(\pi(p))$ $\forall p \in \pi^{-1}U_\alpha$.
Let $\pi(p) = a \in U_{\alpha\beta}$. Then

$$\xi_\beta(p) = \xi_\beta(s_\alpha(a) g_\alpha(p)) = \xi_\beta(s_\beta(a) g_{\beta\alpha}(a) g_\alpha(p)) = \rho(g_{\beta\alpha}(a) g_\alpha(p))^{-1} \circ \sigma_\beta(a)$$
$$= \rho(g_\alpha(p))^{-1} \circ \rho(g_{\alpha\beta}(a)) \circ \sigma_\beta(a) = \rho(g_\alpha(p))^{-1} \circ \sigma_\alpha(a)$$
$$= \rho(g_\alpha(p))^{-1} \circ \xi_\alpha(s_\alpha(a)) = \xi_\alpha(s_\alpha(a) \cdot g_\alpha(p)) = \xi_\alpha(p).$$

Therefore the $\{\xi_\alpha\}$ thus defined glue to define a function $\xi : P \to V$, which obeys $r_g^* \xi = \rho(g)^{-1} \circ \xi$ by definition. If $f \in C^\infty(M)$, then $f\sigma \in \Gamma(E)$ and $(f\sigma)_\alpha = f\sigma_\alpha$ since Ψ_α is fibrewise linear. Then by definition, $\rho(g_\alpha(p))^{-1} \circ \pi^*(f\sigma_\alpha) = \rho(g_\alpha(p))^{-1} \circ \pi^* f\, \pi^* \sigma_\alpha = \pi^* f\, \rho(g_\alpha(p))^{-1} \circ \pi^* \sigma_\alpha$
$= \pi^* f\, \xi_\alpha(p)$

so that the map $\Gamma(E) \to C^\infty_G(P, V)$ thus defined is $C^\infty(M)$-linear. (It is clearly $\mathbb{R}$-linear by defn.)

Conversely, given a G-equivariant $\xi : P \to F$, we define $\sigma \in \Gamma(E)$ as follows. Let $s_\alpha : U_\alpha \to P$ be the canonical local sections. Then let $\sigma_\alpha = s_\alpha^* \xi$. Let $a \in U_{\alpha\beta}$, so that

$$\sigma_\beta(a) = \xi(s_\beta(a)) = \xi(s_\alpha(a) g_{\alpha\beta}(a)) = \rho(g_{\alpha\beta}(a))^{-1} \cdot \xi(s_\alpha(a)) = \rho(g_{\beta\alpha}(a)) \cdot \sigma_\alpha(a).$$

Notice that $s_\alpha^*(\pi^* f\, \xi) = (\pi \circ s_\alpha)^* f\, s_\alpha^* \xi = f\, s_\alpha^* \xi$, so also $C^\infty(M)$-linear ■

20. Examples

(1) Let ω, ω' be connection one-forms for Ehresmann connections $\mathcal{H}, \mathcal{H}'$ on $P \to M$. Then $r_g^* \omega = \mathrm{Ad}_{g^{-1}} \circ \omega$ & similarly for ω'. Now if ξ is vertical, $\omega(\xi) = \omega'(\xi)$ and hence $\tau := \omega - \omega' \in \Omega^1(P; \mathfrak{g})$ is horizontal (ie: $\tau(\xi) = 0$ for ξ vertical).

Let $\tau_\alpha = s_\alpha^* \tau \in \Omega^1(U_\alpha; \mathfrak{g})$. Then $\tau_\alpha = s_\alpha^*\omega - s_\alpha^*\omega' = A_\alpha - A'_\alpha$.

On $U_{\alpha\beta}$, $A_\alpha = \mathrm{Ad}_{g_{\alpha\beta}} \circ A_\beta + g_{\beta\alpha}^* \theta$ and $A'_\alpha = \mathrm{Ad}_{g_{\alpha\beta}} \circ A'_\beta + g_{\beta\alpha}^* \theta$

$\Rightarrow$ $\tau_\alpha = \mathrm{Ad}_{g_{\alpha\beta}} \circ \tau_\beta$. This says that $\{\tau_\alpha\}$ defines $\tau \in \Omega^1(M; \mathrm{ad}\, P)$ where $\mathrm{ad}\, P := P \times_G \mathfrak{g}$.

(2) $H \subset G$ closed and $M = G/H$. Then $G \xrightarrow{\pi} M$ is a ppal H-bundle. Let $\rho : H \to GL(V)$ be a rep$^{\underline{n}}$. Then $E := G \times_H V \to M$ is a homogeneous vector bundle. Then $\Gamma(E) \cong \{ f : G \to V \mid f(p \cdot h) = \rho(h)^{-1} \cdot f(p) \}$ where $\cong$ is one of $C^\infty(M)$-modules. On $\Gamma(E)$ we have a rep. of G: $(g \cdot f)(g_1) := f(g^{-1} g_1)$. This is the rep of G induced by the rep. V of H.

There is a sort of converse of the "associated VB" construction. If $E \xrightarrow{\pi} M$ is a vector bundle (real, rank k, say) we may associate with it a principal $GL(k,\mathbb{R})$-bundle as follows. We can do it in one of two ways. Firstly, we can do it via a local description. Let $\{(U_\alpha, \psi_\alpha)\}_{\alpha \in A}$ be a trivialising atlas for E, with $\psi_\alpha : \pi^{-1} U_\alpha \to U_\alpha \times \mathbb{R}^k$ and transition functions $g_{\alpha\beta} : U_{\alpha\beta} \to GL(k,\mathbb{R})$. We may use the same transition functions to glue $U_\alpha \times GL(k,\mathbb{R})$ and $U_\beta \times GL(k,\mathbb{R})$ along $U_{\alpha\beta}$: if $a \in U_{\alpha\beta}$, then $(a, A) \sim (a, g_{\alpha\beta}(a) A)$ which is equivariant under right multiplication by $GL(k,\mathbb{R})$. The resulting principal $GL(k,\mathbb{R})$ bundle is denoted $GL(E) \xrightarrow{\varpi} M$ and it follows that $E \to M$ is the vector bundle associated to $GL(E)$ via the identity (or 'defining') representation of $GL(k,\mathbb{R})$.

The PFB $GL(E) \xrightarrow{\varpi} M$ can also be understood as the bundle of frames of $E \xrightarrow{\pi} M$. Let $GL(E)_a = \{$ordered bases for $E_a\}$. Let $u = (u_1, \dots, u_k)$ be a frame for E_a. Then $\varpi(u) = a$ defines $\varpi : GL(E) \to M$. If $A \in GL(k,\mathbb{R})$, $u \cdot A$ defined by $(u \cdot A)_i = \sum_j u_j A_{ji}$ is another frame for E_a. Given frames u, u' for E_a, $\exists!$

$A \in GL(k,\mathbb{R})$ such that $u' = u \cdot A$. Let (U, ψ) be a local trivialisation for E. We define a reference frame $\bar{u}(a)$ for each $a \in U$, by $\psi(\bar{u}_i(a)) = (a, e_i)$. ← std. basis for $\mathbb{R}^k$

This defines a trivialisation $\Psi : \varpi^{-1}U \to U \times GL(k,\mathbb{R})$ by $\Psi(u) = (a, \bar{u}(a) \cdot A(u))$ where u is a frame for E_a and $A(u)$ is the unique element in $GL(k,\mathbb{R})$ sending the reference frame $\bar{u}(a)$ to u. Let $B \in GL(k,\mathbb{R})$. Then $\Psi(u \cdot B) = (a, A(u \cdot B))$ where $\bar{u}(a) A(u \cdot B) = u \cdot B = (\bar{u}(a) \cdot A(u)) \cdot B$ $\therefore$ $A(u \cdot B) = A(u) \cdot B$ showing that Ψ is $GL(k,\mathbb{R})$-equivariant. Let $\{(U_\alpha, \Psi_\alpha)\}_{\alpha \in A}$ denote the resulting trivialising atlas. Then if $a \in U_{\alpha\beta}$ and u is a frame for E_a, then $\Psi_\alpha(u) = (a, A_\alpha(u))$ where $\bar{u}_\alpha(a) \cdot A_\alpha(u) = u$. How are $A_\alpha(u)$ and $A_\beta(u)$ related? How are $\bar{u}_\alpha(a)$ and $\bar{u}_\beta(a)$ related?

$$\bar{u}_\beta(a)_i = \psi_\beta^{-1}(a, e_i) = \psi_\alpha^{-1} \circ \psi_\alpha \circ \psi_\beta^{-1}(a, e_i) = \psi_\alpha^{-1}(a, g_{\alpha\beta}(a) e_i)$$
$$= \psi_\alpha^{-1}(a, \sum_j e_j (g_{\alpha\beta}(a))_{ji})$$
$$= \sum_j \psi_\alpha^{-1}(a, e_j)\, g_{\alpha\beta}(a)_{ji} = \sum_j \bar{u}_\alpha(a)_j\, g_{\alpha\beta}(a)_{ji}$$

ie: $\bar{u}_\beta(a) = \bar{u}_\alpha(a) \cdot g_{\alpha\beta}(a)$

$\therefore$ $\bar{u}_\alpha(a) \cdot A_\alpha(u) = u = \bar{u}_\beta(a) \cdot A_\beta(a) = \bar{u}_\alpha \cdot g_{\alpha\beta}(a) \cdot A_\beta(a) \implies A_\alpha(u) = g_{\alpha\beta}(a) \cdot A_\beta(a)$

agreeing with the local description.

21. Definition Let $E \xrightarrow{\pi} M$ be a vector bundle. By a Koszul connection on E we mean an $\mathbb{R}$-bilinear map

$$\nabla : \mathfrak{X}(M) \times \Gamma(E) \longrightarrow \Gamma(E)$$
$$(X, s) \longmapsto \nabla_X s$$

satisfying ① $\nabla_{fX} s = f \nabla_X s$

and ② $\nabla_X (fs) = X(f) s + f \nabla_X s$, $\forall f \in C^\infty(M), X \in \mathfrak{X}(M), s \in \Gamma(E)$.

Suppose that $E = P \times_G V$ for some G-PFB $P \xrightarrow{\pi} M$. Then an Ehresmann connection on P induces a Koszul connection on E. For this it is convenient to use the $C^\infty(M)$-module isomorphism $\Gamma(E) \cong C^\infty_G(P,V)$ and we will define ∇ on $C^\infty_G(P,V)$.

Let $\mathcal{H} \subset TP$ be an Ehresmann connection: $T_pP = \mathcal{H}_p \oplus \ker(\pi_*)_p$ & $(r_g)_*\mathcal{H}_p = \mathcal{H}_{p\cdot g}$. We define $h : T_pP \to T_pP$ to be the projector onto $\mathcal{H}$ along $\ker \pi_*$: if we write $\xi \in T_pP$ as $\xi^h + \xi^v$ where $\xi^h \in \mathcal{H}_p$ & $\pi_*(\xi^v) = 0$, $h(\xi) = \xi^h$. Let $h^* : T_p^*P \to T_p^*P$ denote the dual map, so $(h^*\alpha)(\xi) = \alpha(h(\xi))$. Caveat: $h^* \circ d \neq d \circ h^*$ (not a pullback despite notation)

Let $X \in \mathfrak{X}(M)$. Then given $p \in P_a$, let $\xi \in T_pP$ be such that $\pi_*\xi = X(a)$. We define $\nabla_X \psi|_p := (d\psi)_p(h\xi)$ or $d^\nabla\psi = h^* d\psi$. This is well-defined because if $\pi_*\xi = \pi_*\xi'$ then $h\xi = h\xi'$. Also $\nabla_X\psi \in C^\infty_G(P,V)$, because the split $TP = \mathcal{V} \oplus \mathcal{H}$ is G-invariant and hence $r_g^* h^* = h^* r_g^*$. So that

$$r_g^* d^\nabla \psi = r_g^* h^* d\psi = h^* r_g^* d\psi = h^* d r_g^* \psi = h^* d(\rho(g)^{-1} \circ \psi) = \rho(g)^{-1} \circ h^* d\psi = \rho(g)^{-1} d^\nabla \psi.$$

22. Proposition ∇ defines a Koszul connection on E.

Proof We check the two conditions.

① $\nabla_{fX}\psi = d\psi(h(f\cdot\xi)) = d\psi(h(\pi^*f\,\xi)) = \pi^*f\, d\psi(h\xi) = f\cdot\nabla_X\psi$

② $\nabla_X(f\cdot\psi) = \nabla_X(\pi^*f\,\psi) = d(\pi^*f\,\psi)(h\xi) = (\pi^*df)(h\xi)\,\psi + \pi^*f\,\nabla_X\psi$
$= \pi^*(df(\pi_*h\xi))\,\psi + f\cdot\nabla_X\psi = \pi^*(df(\pi_*\xi))\,\psi + f\cdot\nabla_X\psi$
$= \pi^*(Xf)\,\psi + f\cdot\nabla_X\psi = Xf\cdot\psi + f\cdot\nabla_X\psi$. ■

We will now derive a more calculationally useful formula for the Koszul connection of $P\times_G V$ induced by the Ehresmann connection on P.

Let $\psi \in C^\infty_G(P,V)$ and let $\xi \in \mathfrak{X}(P)$. We decompose $\xi = h\xi + \xi^v$ where $\pi_*\xi^v = 0$. Then $d\psi(h\xi) = d\psi(\xi - \xi^v) = d\psi(\xi) - d\psi(\xi^v)$.

The derivative $\xi^v\psi$ only depends on the value of ξ^v at a point, so we can take ξ^v to be the fundamental vector field $\xi_{\omega(\xi^v)} = \xi_{\omega(\xi)}$ corresponding to the G-action. Therefore,

$$\xi^v\psi = \xi_{\omega(\xi)}\psi = \frac{d}{dt}\left(\psi \circ r_{\exp(t\omega(\xi))}\right)\Big|_{t=0} = \frac{d}{dt}\,\rho(\exp(-t\omega(\xi)))\circ\psi\,\Big|_{t=0}$$
$$= -\rho(\omega(\xi))\circ\psi$$

Therefore $(d\psi)(h\xi) = d\psi(\xi) + \rho(\omega(\xi))\circ\psi$ or, abstracting ξ,

$$\boxed{d^\nabla\psi = d\psi + \rho(\omega)\circ\psi}$$

Finally, we give a formula for $\nabla_X \sigma$ where $\sigma \in \Gamma(P \times_G V)$ now viewed as a family $\{\sigma_\alpha : U_\alpha \to V\}$ of functions transforming in overlaps as $\sigma_\alpha(a) = \rho(g_{\alpha\beta}(a))\,\sigma_\beta(a) \quad \forall a \in U_{\alpha\beta}$.

$$
\begin{aligned}
d^\nabla \sigma_\alpha &= d^\nabla s_\alpha^* \psi = d^\nabla(\psi \circ s_\alpha) = d(\psi \circ s_\alpha) \circ h \\
&= d(s_\alpha^* \psi) \circ h = s_\alpha^*(d\psi) \circ h \\
&= s_\alpha^* d^\nabla \psi = s_\alpha^* \left(d\psi + \rho(\omega) \circ \psi \right) \\
&= d s_\alpha^* \psi + \rho(s_\alpha^* \omega) \circ s_\alpha^* \psi \\
&= d\sigma_\alpha + \rho(A_\alpha) \cdot \sigma_\alpha
\end{aligned}
$$

Hence if $X \in \mathfrak{X}(M)$,

$$\nabla_X \sigma_\alpha := X(\sigma_\alpha) + \rho(A_\alpha(X)) \cdot \sigma_\alpha$$

The following exercise justifies the name covariant derivative

Exercise Show that $\nabla_X \sigma_\alpha$ transforms like σ_α on overlaps; that is, on $U_{\alpha\beta}$,

$$\nabla_X \sigma_\alpha = \rho(g_{\alpha\beta}) \cdot \nabla_X \sigma_\beta .$$

In summary, given a G-PFB $P \to M$ and a (finite-dim'l) representation $\rho : G \to GL(V)$, we constructed a VB $P \times_G V \to M$. Every VB is obtained in this way from its frame bundle. We then introduced the notion of a Koszul connection on a VB and showed that an Ehresmann connection on P induces a Koszul connection on $P \times_G V$. The converse is also true: a Koszul connection on E induces an Ehresmann connection on $GL(E)$. You may wish to think about how to prove that, but here's a hint: recall how we built ω from the gauge fields. So given ∇, mimic that construction to obtain ω_α from the gauge fields A_α for ∇ ($\rho = \mathrm{id}$) and show $\omega_\alpha = \omega_\beta$ on $U_{\alpha\beta}$.

Remarks after the lecture

Carlos asked about re-interpreting the Ehresmann connection along the lines of the interpretation of a Koszul connection as a splitting of the sequence

$$\cdots \to \mathrm{Der}(E) \xrightarrow{\sigma} \mathfrak{X}(M) \to 0$$

$$\nabla$$

which suggests the following. If $P \to M$ is the frame bundle of E, then ∇ induces an Ehresmann connection on P, which is a splitting of

$$0 \to \mathcal{V} \to TP \xrightarrow{T\pi} TM \to 0$$

$$\mathcal{H}$$

This would seem to suggest that $\mathrm{Der}(E)$ is related to $\mathfrak{X}(P)$?

Carlos remarked that the associated VB construction could be viewed as starting from the trivial bundle $P\times V \to P$ over P and lifting the G-action to $P\times V$ as in the lecture and then taking the quotient bundle:

$$\begin{array}{ccccc} P\times V & & (P\times V)/G & = & P\times_G V \\ \downarrow & \longrightarrow & \downarrow & & \downarrow \\ P & & P/G & = & M \end{array}$$

What is it about the G-action which guarantees that the quotient will be a bundle? This observation says that sections of $P\times_G V$ are G-invariant sections of the trivial bundle $P\times V \to P$, i.e. G-invariant functions $P \to V$.

Guido also observed that one could show the iso $\Gamma(E) \cong C^\infty_G(P,V)$ more directly. A function $\xi \in C^\infty_G(P,V)$ defines $\sigma: M \to (P\times V)/G$ by $\sigma(a) = [(p,\xi(p))]$ for any $p \in \pi^{-1}(a)$. This does not depend on p because $[(p\cdot g, \xi(pg))] = [(p\cdot g, \rho(g)^{-1}\cdot \xi(p))] = [(p,\xi(p))]$.
The inverse map $\sigma \mapsto \xi$ seems to require a trivialisation, since we need to identify the fibres of E with V. Is that really the case?